INSTABILITY OF THE EQUILIBRIUM OF FLUIDS

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This paper examines the problem of inversion of the Lagrange theorem in hydrodynamics. The essence of the problem is proving the instability of the position of equilibrium (rest) of a mechanical system in the absence of a potential energy minimum in the system [1-4]. Linear problems of stability of the equilibrium of ideal incompressible and compressible fluids are examined, and allowance is made for factors such as capillarity, density or entropy stratification, and rotation. The result consists of a priori estimates of the solutions of these problems which provide evidence of the growth over time of the functional M — the mean square of the Lagrangian displacements of the fluid particles. The character of the increase in M is exponential, with an increment which is simply calculated from the initial data of the problem. To illustrate the degree of generality of the approach being used, we make an estimate of the increase in M in a problem concerning the stability of the equilibrium of an anisotropic elastic body.

The proposed method of obtaining results on instability is a variant of the direct method of Lyapunov. The principle difficulty of the latter is finding the specific form of the Lyapunov functional — which increases by virtue of the equations of motion of the system. The functional M used in the present study was introduced in [4] for problems concerning the stability of bodies with fluid-containing cavities and for elastic bodies in [5, 6]. The authors of [4-6] obtained the estimate $dM/dt > ct$ with the constant $c > 0$.

1. Ideal Incompressible Stratified Capillary Fluid. We are examining three-dimensional motions of an ideal incompressible fluid of nonuniform density. The motions take place in an external body-force field. The fluid as a whole occupies the region $\tau$ with a fixed boundary $\partial\tau$. The fluid surface $\Gamma$ divided the region $\tau$ into two parts: $\tau^+$ and $\tau^-$. In each of these parts, the field of density $\rho$ is continuous, while on $\Gamma$ itself there is a density discontinuity $[\rho] \equiv \rho^+ - \rho^-$. Motion in the region $\tau^\pm$ is described by the equations

$$\rho Du_i = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \Phi}{\partial x_i},\ D\rho = 0,\ \ \frac{\partial u_k}{\partial x_k} = 0,\ D \equiv \frac{\partial}{\partial t} + u_k\frac{\partial}{\partial x_k}, \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ are Cartesian coordinates; $\mathbf{u} = (u_1, u_2, u_3)$ and p are the velocity and pressure fields; the signs $\pm$ of the sought functions will be dropped; $\Phi = \Phi(\mathbf{x})$ is the potential of the external field of body forces. Summation is performed over repeating vector indices. The following impermeability condition is imposed on $\partial\tau$:

$$\mathbf{u}\cdot\mathbf{n} = 0 \tag{1.2}$$

($\mathbf{n}$ is an external normal to $\partial\tau$). On the fluid surface $\Gamma$, given by the equation $F(x, t) = 0$, the following kinematic and dynamic conditions are satisfied:

$$dF/dt = 0,\ [p] \equiv p^+ - p^- = -\sigma(k_1 + k_2). \tag{1.3}$$

Here, $\sigma$ is the constant value of surface tension; $k_1$ and $k_2$ are the curvatures of the principal normal sections of the surface $\Gamma$ (each of which is assumed to be positive if the corresponding normal section is convex in the direction of the region $\tau^+$). On the line $\gamma$ of intersection of the surfaces $\partial\tau$ and $\Gamma$, we impose the Dupre–Young condition [4, 7]

$$\sigma\cos\alpha = \sigma^- - \sigma^+ \tag{1.4}$$

where $\alpha$ is the contact angle; $\sigma^-$ and $\sigma^+$ are the constant values of surface tension on $\partial\tau^+$ and $\partial\tau^-$. We use $\partial\tau^\pm$ to designate those parts on which the curve $\gamma$ divides the solid surface $\partial\tau$. The initial data for (1.1)-(1.4) is given in the form

$$\rho(\mathbf{x}, 0) = \rho^0(\mathbf{x}),\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}),\ F(\mathbf{x}, 0) = F^0(\mathbf{x}) \tag{1.5}$$

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with obvious restrictions on the functions $\mathbf{u}^0(\mathbf{x})$ and $F^0(\mathbf{x})$. All of the functions used here are assumed to be continuous, as are their derivatives in the equations of motion and the boundary conditions. Energy is conserved in the solutions of problem (1.1)-(1.5)

$$
\begin{gathered}
dE_1/dt = 0,\ E_1 = K_1 + \Pi_1 = \text{const}, \\
2K_1 \equiv \int_\tau \rho u_i u_i d\tau,\ d\tau \equiv dx_1 dx_2 dx_3, \\
\Pi_1 \equiv \int_\tau \rho\Phi d\tau + \sigma|\Gamma| + \sigma^+|\partial\tau^+| + \sigma^-|\partial\tau^-|
\end{gathered} \tag{1.6}
$$

($|\Gamma|$ and $|d\tau^\pm|$ are the areas of the corresponding surfaces).

The states of hydrostatic equilibrium are the solutions of problem (1.1)-(1.5) in the form

$$
\begin{gathered}
\left.\begin{array}{l}\mathbf{u} \equiv 0,\ \rho = \rho_0(\mathbf{x}),\ p = p_0(\mathbf{x}), \\ \nabla p_0 = -\rho_0\nabla\Phi,\ \nabla\rho_0 \times \nabla\Phi = 0\end{array}\right\} \text{ in } \tau_0^\pm, \\
[p_0] = -\sigma(k_1 + k_2),\ [\rho_0] \neq 0 \text{ on } \Gamma_0
\end{gathered} \tag{1.7}
$$

($\Gamma_0$ is the equilibrium surface of the density discontinuity separating the region $\tau$ into the parts $\tau_0^\pm$).

Linearization of the relations of problem (1.1)-(1.5) for solution (1.7) gives

$$
\begin{gathered}
\left.\begin{array}{l}\rho_0 u_{it} = -\dfrac{\partial p}{\partial x_i} - \rho\dfrac{\partial\Phi}{\partial x_i}, \\ \rho_t + u_k\dfrac{\partial\rho_0}{\partial x_k} = 0,\ \dfrac{\partial u_k}{\partial x_k} = 0\end{array}\right\} \text{ in } \tau, \\
\left.\begin{array}{l}[\mathbf{u}\cdot\mathbf{v}] = 0,\ N_t = \mathbf{v}\cdot\mathbf{u}, \\ [p] = \sigma(aN - \Delta N),\ a \equiv \dfrac{[\rho_0]}{\sigma}\dfrac{\partial\Phi}{\partial\nu} - k_1^2 - k_2^2\end{array}\right\} \text{ on } \Gamma_0, \\
\frac{\partial N}{\partial e} + \chi N = 0,\ \chi \equiv \frac{k\cos\alpha - \bar{k}}{\sin\alpha} \text{ on } \gamma_0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\tau.
\end{gathered} \tag{1.8}
$$

Here, $\mathbf{u}$, $p$, and $\rho$ are fields of perturbations of velocity, pressure, and density, with the signs $\pm$ and primes [distinguishing the perturbation fields from the complete solutions (1.1)] omitted; $\nu$ is a unit normal to $\Gamma_0$, directed from $\tau_+$ to $\tau_-$; $\partial\Phi/\partial\nu \equiv \nu\cdot\nabla\Phi$; N is the displacement of the fluid surface $\Gamma$ along the normal to $\Gamma_0$; $\Delta$ is Beltrami's second differential parameter [8, p. 190]; $k$ and $\bar{k}$ are the curvatures of the normal sections of the surface $\Gamma_0$ and $\partial\tau$ along the directions $e$ and $e_1$; the unit vectors $e$ and $e_1$, in turn, are normal to the unperturbed position of the wetting line $\gamma_0$ and lie in planes tangent to the surfaces $\Gamma_0$ (in the direction from $\Gamma_0$) and to $\partial\tau$ (in the direction from $\partial\tau_0^+$), respectively; $\partial N/\partial e$ is the derivative of N in the direction e. When $\rho_0^+ = \text{const}$, $\rho_0^- \equiv 0$, Eqs. (1.8) coincide with the relations presented in [7]. In a linear approximation, initial data (1.5) reduces to the form

$$
\rho(\mathbf{x}, 0) = \rho^1(\mathbf{x}),\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}),\ N(\mathbf{x}, 0) = N^0(\mathbf{x}). \tag{1.9}
$$

The following analog of the energy integral is valid for solutions of initial boundary-value problem (1.8)-(1.9)

$$
\begin{gathered}
E \equiv K + \Pi = \text{const},\quad \Pi \equiv \Pi_\rho + \Pi_\sigma, \\
2K \equiv \int_\tau \rho_0 u_i u_i d\tau,\quad 2\Pi_\rho \equiv -\int_\tau \Phi'(\rho_0)\rho^2 d\tau, \\
\frac{2\Pi_\sigma}{\sigma} \equiv \int_{\Gamma_0} \{aN^2 + \nabla(N, N)\}\, dS + \int_{\gamma_0} \chi N^2 dl,
\end{gathered} \tag{1.10}
$$

in which $\nabla(N, N)$ is Beltrami's first differential parameter [8]; the derivatives $\Phi'(\rho_0) \equiv d\Phi/d\rho_0 = \nabla\Phi/\nabla\rho_0$ has the meaning in the regions $\tau_0^\pm$ by virtue of equilibrium conditions (1.7) if $\nabla\rho_0 \neq 0$. By the integral over $\tau$ in (1.10) we mean the sum of the integrals over $\tau_0^+$ and $\tau_0^-$. It should be noted that the expression for $\Pi_\sigma$ coincides with that presented in [7, 9, 10].

If $\nabla\rho_0 = 0$ somewhere in $\tau_0^{\pm}$, then the functional $\Pi_\rho$ in (1.10) loses meaning. To exclude the singularity that would occur in this case in the integrand, we must restrict ourselves to examining a narrower class of motions in which the perturbations of fluid-particle density (the Lagrangian of the density perturbation) are equal to zero. In other words, the density of each fluid particle does not change during perturbations. The perturbations consist only of displacements of the particles from the equilibrium position. This class of motions is most simply described by means of the Lagrangian displacements $\xi(x, t)$ [11], for which the following relations are satisfied:

$$\xi_t = \mathbf{u}, \quad \operatorname{div}\xi = 0, \quad \rho = -(\xi\cdot\nabla)\rho_0 \text{ in } \tau_0^{\pm},$$
$$N = \xi\cdot\mathbf{v}, \quad [\xi\cdot\mathbf{v}] = 0 \text{ on } \Gamma_0, \tag{1.11}$$
$$\xi\cdot\mathbf{n} = 0 \text{ on } \partial\tau.$$

All of the equalities (1.8) in the terms $\xi$ are rewritten in an obvious manner. The initial data (1.9) are replaced for (1.8), (1.11) by the following:

$$\xi(\mathbf{x}, 0) = \xi^0(\mathbf{x}), \ \mathbf{u}(\mathbf{x}, 0) = \xi_t(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}). \tag{1.12}$$

The functional $\Pi_\rho$ from (1.10) for this class of motions takes the form

$$2\Pi_\rho = -\int\limits_\tau \frac{\partial\Phi}{\partial x_i}\frac{\partial\rho_0}{\partial x_k}\xi_i\xi_k d\tau = \int\limits_\tau bN_1^2 d\tau,$$
$$b \equiv -\nabla\rho_0\cdot\nabla\Phi, \ N_1 \equiv \xi\cdot\nabla\Phi/|\nabla\Phi|, \tag{1.13}$$

while the functionals K and $\Pi_\sigma$ (1.10) remain as before.

If the condition $\Pi \geq 0$ is satisfied for all fields $\xi$, then stability by the linear approximation follows from the equality $E(t) = E(0)$ (1.10). In this approach, it is necessary to determine stability in regard to some of the variables [12]. The reason for this is that E includes only the components of the displacements N, $N_1$, and $\mathbf{v}\times\nabla N$, not the complete vector $\xi$. For an ideal fluid, no results have yet been obtained on stability in the norm of any definite function space $\xi(x)$.

Confirmation of the stability of state (1.7) at $\Pi \geq 0$ comes from one of the forms of the Lagrangian theorem [1-7] which links the fact of the stability of the state of rest with the presence of a potential energy minimum in this state. Indeed, as in [4, 7, 9, 10], it can be shown that the second variation $\delta^2\Pi_1$ of potential energy (1.6), written in the appropriate notation, coincides with the functional $\Pi$ (1.10), (1.13). Here, $\delta\Pi_1 = 0$ by virtue of equilibrium conditions (1.7). Simultaneously with stability (1.7), by the linear approximation we can also expect nonlinear stability. However, its determination requires special definitions and proofs [1, 4, 7, 13-15].

2. Direct Lyapunov Method in the Proof of Instability. The goal of the subsequent exposition is to obtain an inversion of the Lagrangian theorem, i.e., to prove the instability of the state of equilibrium (1.7) in the absence of a minimum of potential energy $\Pi_1$ (1.6) or $\Pi$ (1.10), (1.13) in this state. In terms of Lagrangian displacements $\xi$, this means that there exists a field $\xi = \xi^*(x)$ for which

$$\Pi = \Pi^* < 0 \text{ at } \xi = \xi^*(\mathbf{x}). \tag{2.1}$$

For other fields $\xi(x)$, inequality (2.1) can be replaced by its opposite, i.e., state (1.7) is an infinite-dimensional analog of the "saddle" point of functionals $\Pi_1$ and $\Pi$. Condition (2.1) is satisfied if $\Pi_\rho < 0$ and (or) $\Pi_\sigma < 0$ (1.10), (1.13). The inequality $\Pi_\rho < 0$ means that the density $\rho_0$ increases "upward" somewhere in $\tau$ ($\nabla\rho_0\cdot\nabla\Phi > 0$), while $\Pi_\sigma < 0$ reflects the more complex balance of the effects of surface tension and the density discontinuity on $\Gamma_0$. As is known [7], even in a situation where a heavier fluid is "on top" ($[\rho_0]\mathbf{v}\cdot\nabla\Phi < 0$), the functional $\Pi_\sigma$ can be positively determined. Conversely, it is possible to have cases in which the heavy fluid is below $[\rho_0]\mathbf{v}\cdot\nabla\Phi > 0]$. For this case, we have perturbations with $\Pi_\sigma < 0$.

To demonstrate instability, we introduce the functionals

$$M \equiv \int_\tau \rho_0 \xi_i \xi_i d\tau, \quad W \equiv \dot{M}/2 = \int_\tau \rho_0 u_i \xi_i d\tau, \tag{2.2}$$

where the superimposed dot denotes a derivative with respect to time. The integrals over $\tau$, as previously, mean the sum of the integrals over $\tau_0^{\pm}$. Functionals of the type (2.2) were first used to prove the instability of solids with fluid-containing cavities in [4]. Using (1.8)-(1.13), we can obtain

$$\ddot{M} = 2\dot{W} = 4(K - \Pi) = 8K - 4E. \tag{2.3}$$

In deriving (2.3), we also used the generalized Green formula [8, p. 192]. It follows from the Cauchy–Bunyakovskii inequality [16] that $\dot{M}^2 = 4W^2 \le 8KM$. Using (2.3) to exclude 8K from this expression, we have $M\ddot{M} - \dot{M}^2 + 4EM \ge 0$. After this inequality is divided by $M^2$, it takes the form

$$\frac{d}{dt}\left(\frac{\dot{M}}{M}\right) \geqslant -\frac{4E}{M}. \tag{2.4}$$

For any perturbation with $E < 0$, it follows from (2.4) that

$$\frac{d}{dt}\left(\frac{\dot{M}}{M}\right) > 0, \quad \frac{\dot{M}}{M} > \frac{2W(0)}{M(0)} \equiv 2\lambda, \tag{2.5}$$

after which selection of $\lambda > 0$ leads to the sought estimate of the increase in perturbation

$$M(t) > M(0)\exp(2\lambda t). \tag{2.6}$$

The above choice $E < 0$ and $\lambda > 0$ is possible by virtue of (2.1) and because the fields $\xi$ and $\mathbf{u}$ at the initial moment of time are given by the inequality (1.13). In fact, with allowance for (1.10), (2.1), the inequality $E < 0$ means that the initial data (1.12) is chosen so that $K(0) < |\Pi^*|$. Then the initial data are used to calculate $\lambda \equiv W(0)/M(0)$. If it turns out that $\lambda < 0$, then in (1.12) we need to change the sign of one of the functions $\xi^0(x)$ or $\mathbf{u}^0(x)$ in (1.12) and leave the sign of the other function unchanged. Thus, the construction ensures the existence of initial data corresponding to (2.6) with $\lambda > 0$. It is understood that since we are not considering mathematical questions regarding the existence of solutions, inequality (2.6) has the character of an a priori estimate.

The upper bound of $\lambda$ is evaluated by means of the Cauchy–Bunyakovskii inequality:

$$\lambda \leqslant \sqrt{2K(0)M(0)}/M(0) = \sqrt{2K(0)/M(0)} < \sqrt{-2\Pi(0)/M(0)}. \tag{2.7}$$

The last relation in (2.7) arises from the requirement $E < 0$. To refine estimate (2.7), it would be useful to examine a class of initial data, narrower than (1.12), that contains the function $\xi^*(x)$ (2.1) and the constant k:

$$\xi(\mathbf{x}, 0) = \xi^*(\mathbf{x}), \ \mathbf{u}(\mathbf{x}, 0) = \xi_t(\mathbf{x}, 0) = k\xi^*(\mathbf{x}). \tag{2.8}$$

For these data, the conditions $E < 0$ and $\lambda > 0$ take the form

$$E = k^2 M(0)/2 + \Pi(0) < 0, \ k = \lambda > 0,$$

which in turn leads to

$$0 < k = \lambda < \sqrt{-2\Pi(0)/M(0)}. \tag{2.9}$$

It is evident that the interval (2.9) of possible values of the increment $\lambda$ coincides with estimate (2.7), i.e., the latter gives the exact boundary of the interval of the increments.

The largest increment $\lambda_{max}$ in (2.7), (2.8) can be calculated by solving the variational problem of determining the maximum of the quantity $\lambda = \sqrt{-2\Pi/M}$. The allowable fields $\xi(x)$ should satisfy the conditions $\operatorname{div}\xi = 0$ in $\tau_0^{\pm}$, $\xi\cdot\mathbf{n} = 0$ on $\partial\tau$ and $[\xi\cdot\mathbf{v}] = 0$ on $\Gamma_0$. This problem is not solved here. We will instead restrict ourselves to two particular observations regarding it.

1. If $\sigma = 0$ and there is an unstable density discontinuity on $\Gamma_0$ (i.e., $[\rho_0]\mathbf{v}\cdot\nabla\Phi < 0$), then $\lambda_{max} \to \infty$. This means that the problem is ill-conditioned, in accordance with Adamar's definition. In fact, in the ratio $\Pi/M$, the numerator $\Pi$ contains a volume integral and a negative surface integral, while the denominator M contains only a volume integral. Choos-

ing the trial functions $\xi(x)$ to be rapidly oscillating along $\Gamma_0$ and rapidly decaying with increasing distance from $\Gamma_0$, we obtain $M \to 0$ at fixed $\Pi < 0$.

2. If on $\Gamma_0$ we have $\sigma = 0$, $[\rho_0] = 0$, then only continuous stratification remains and it follows from (2.1) and (1.13) that

$$\lambda^2_{\max} < \max_\tau (\nabla\rho_0 \cdot \nabla\Phi) \Big/ \min_\tau \rho_0. \tag{2.10}$$

A noteworthy feature of the resulting instability estimate (2.6) is the fact that the method by which it was obtained is independent of the specific form of potential energy. For (2.6) to be valid, it is necessary only that perturbations exist with negative energy (2.1) and that Eq. (2.3) be satisfied. The latter pertains to the family of relations for the virial and is satisfied for many nondissipative mechanical systems [2, 4, 11, 17]. Such universality of estimate (2.6) makes it possible, without changing its form, to consider the rotation of a medium, its compressibility, and other factors. Examples of generalizations are given below.

<u>3. Instability of a Baroclinic Vortex.</u> We will examine rotationally symmetric motions of an incompressible fluid in an axisymmetric vessel $\tau$. In the cylindrical coordinate system $(r, z, \varphi)$, the components of the velocity field are $(u_1, u_2, u_3)$. The potential of the external body forces $\Phi = \Phi(r, z)$. The density field $\rho$ is continuous. Using the notation $\mu \equiv (ru_3)^2$, $\Psi \equiv 1/2r^2$, we write the equations of motion in the form

$$\begin{gathered} D\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi - \mu\nabla\Psi, \quad D\mu = 0, \quad D\rho = 0, \\ u_{1r} + \frac{u_1}{r} + u_{2z} = 0, \quad D \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \quad \mathbf{u} = (u_1, u_2), \quad \nabla \equiv \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right), \end{gathered} \tag{3.1}$$

these equations being an example of the analogy between the effects of density stratification and rotation [13, Ch. 8]. The baroclinic vortex whose stability is being studied is represented by the exact solution of (3.1) in the form

$$\mathbf{u} \equiv 0, \ \mu = \mu_0(r, z), \ \rho = \rho_0(r, z), \ p = p_0(r, z), \tag{3.2}$$

where the fields $\mu_0$, $\rho_0$, and $p_0$ are continuous together with their first derivatives and are connected by the relation

$$\nabla p_0 + \rho_0\nabla\Phi + \rho_0\mu_0\nabla\Psi = 0. \tag{3.3}$$

Linearization of (3.1) for the solution (3.2) and introduction of the field of Lagrangian displacements $\xi$ (1.11) lead to the problem

$$\left.\begin{aligned} &\rho_0 u_{it} + m_{ij}\xi_j + \frac{\partial p}{\partial x_i} = 0, \\ &\xi_{1r} + \frac{\xi_1}{r} + \xi_{2z} = 0, \quad u_i = \xi_{it}, \\ &m_{ij} \equiv \frac{\partial\Psi}{\partial x_i}\frac{\partial\rho_0\mu_0}{\partial x_j} + \frac{\partial\Phi}{\partial x_i}\frac{\partial\rho_0}{\partial x_j} \end{aligned}\right\} \text{ in } \tau, \quad \xi_i n_i = 0 \text{ on } \partial\tau, \tag{3.4}$$

where all of the vector indices take values of 1 and 2. The matrix $m_{ij}$ is symmetric, which can be demonstrated after application of the operator rot to (3.3). The below energy integral is valid for (3.4):

$$E = K + \Pi = \text{const}, \quad 2K \equiv \int_\tau \rho_0 u_i u_i d\tau, \quad 2\Pi \equiv \int_\tau m_{ij}\xi_i\xi_j d\tau. \tag{3.5}$$

For the functional M (2.2) (with summation from 1 to 2), we use (3.4) and (3.5) to again obtain Eq. (2.3). After this, in the presence of perturbations with $\Pi < 0$, repetition of the steps in Sec. 2 leads to the required estimate (2.6). Thus, with condition (2.1), baroclinic vortex (3.2) is unstable and its perturbations increase exponentially. The estimate of the increment (2.10) is replaced by

$$\lambda^2_{\max} < \max_\tau (-m_1, -m_2) \Big/ \min_\tau \rho_0$$

($m_1$ and $m_2$ are eigenvalues of the matrix $m_{ij}$). A practical criterion of flow instability (3.2) is negativity of at least one of the numbers $m_1$, $m_2$ in any part of $\tau$.

The requirement of smoothness of the fields $\mu_0$ and $\rho_0$ (3.2) is not essential. We follow the same procedure as in Sec. 2 in examining baroclinic vortices with discontinuities $[\rho_0] \neq 0$ and $[\mu_0] \neq 0$ on certain surfaces inside $\tau$ and in allowing for surface tension. With condition (2.1) (and the corresponding expression for $\Pi$), this approach leads to the same estimate (2.6).

The problem of the stability of a baroclinic circular vortex in the class of rotationally symmetrical perturbations was examined previously in connection with applications to atmospheric physics [18-21]. These studies also employed the direct Lyapunov method, while the functional M (2.2) was introduced in [21]. The overall result in [21] consisted of obtaining the estimate $M > ct^4$ (c being a constant).

4. Instability of States of Rest of a Compressible Fluid. We will examine three-dimensional adiabatic motions of an ideal compressible fluid located in the region $\tau$ with the boundary $\partial\tau$. The motions are described by the solutions of the system of equations

$$D\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \quad D\rho + \rho \operatorname{div} \mathbf{u} = 0, \quad Ds = 0, \tag{4.1}$$

augmented by the thermodynamic relations

$$e = e(\rho, s), \; de = Tds - pd(1/\rho) \tag{4.2}$$

and the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\tau, \tag{4.3}$$

where s, T, and e are the fields of entropy, temperature, and internal energy. The remaining notation is the same as was used in (1.1) and (1.2). For the solutions of (4.1)-(4.3), we have the energy integral

$$E = \int\limits_\tau \rho\left(\frac{u_i u_i}{2} + e(\rho, s) + \Phi\right) d\tau = \text{const}. \tag{4.4}$$

We will study the problem of the stability of hydrostatic equilibrium — the exact solution of (4.1)-(4.3) of the form

$$\mathbf{u} = 0, \; \rho = \rho_0(x), \; p = p_0(x), \; s = s_0(x), \tag{4.5}$$

in which the equipotential surfaces of $\rho_0$, $p_0$, $s_0$, and $\Phi$ coincide with one another. All of the fields (4.5) and their first derivatives are assumed to be continuous. Linearization of Eqs. (4.1) for solution (4.5) gives

$$\begin{gathered} \rho_0\mathbf{u}_t = -\nabla p - \rho\nabla\Phi, \; s_t + (\mathbf{u}\cdot\nabla)s_0 = 0, \\ \rho_t + (\mathbf{u}\cdot\nabla)\rho_0 + \rho_0 \operatorname{div} \mathbf{u} = 0 \end{gathered} \tag{4.6}$$

(u, p, $\rho$, and s are perturbations of velocity, pressure, density, and entropy). The boundary conditions for (4.6) have the same form as (4.3). Introduction of the field of Lagrangian displacements $\xi(x, t)$ ($\xi_t \equiv \mathbf{u}$) leads to the replacement of (4.3) and (4.6) by the relations [11]

$$\left.\begin{aligned} &\rho_0\xi_{tt} = -\nabla p - \rho\nabla\Phi, \quad s = -(\xi\cdot\nabla)s_0, \\ &\rho = -(\xi\cdot\nabla)\rho_0 - \rho_0 \operatorname{div}\xi \end{aligned}\right\} \text{ in } \tau, \quad \xi\cdot\mathbf{n} = 0 \text{ on } \partial\tau. \tag{4.7}$$

The energy integral for (4.7) is

$$\begin{gathered} E \equiv K + \Pi + A = \text{const}, \\ 2\Pi \equiv \int\limits_\tau \rho_0\Omega^2 N_1^2 d\tau, \quad \Omega^2 \equiv -\frac{\nabla\Phi\nabla\rho_0}{\rho_0} - \left(\frac{\nabla\Phi}{c_0}\right)^2, \\ 2A \equiv \int\limits_\tau \frac{p^2}{\rho_0 c_0^2} d\tau, \quad p \equiv c_0^2\rho + \rho_0^2 e_{\rho s}s. \end{gathered} \tag{4.8}$$

Here, K and $N_1$ are taken from (1.10) and (1.13); $\Omega$ is the buoyancy frequency (Brunt–Väsäilä); $c_0$ is the speed of sound. The terms K, $\Pi$, and A are interpreted as the kinetic, potential, and acoustic parts of the total energy E of small perturbations [22].

Stability (4.5), at $\Omega^2 > 0$, $c_0{}^2 > 0$, $e_{\rho s} > 0$, is also one form of the Lagrangian theorem on the stability of the equilibrium of a mechanical system in the presence of a potential energy minimum. Direct calculations can show that the first variation of potential energy from (4.4) for the state (4.5) is equal to zero, while the second variation coincides with $\Pi + A$ from (4.8).

Now let there be a field of Lagrangian displacements $\xi^*(x)$ for which

$$\Pi + A = \Pi^* + A^* < 0 \text{ at } \xi = \xi^*(x). \tag{4.9}$$

In terms of the local properties of equilibrium (4.5), this means that at least one of the following three inequalities exists in some part of $\tau$: $\Omega^2 < 0$, $c_0{}^2 < 0$, $e_{\rho s} < 0$. The first of these inequalities corresponds to unstable stratification, while the last two reflect certain anomalous properties of the equation of state.

For the functional M (2.2), we use (4.7)-(4.9) to obtain the analog of Eq. (2.3), $\dot{M} = 4(K - \Pi - A) = 8K - 4E$. We then use (4.9) and repeat the procedures in Sec. 2 to again obtain estimate (2.6).

Application of the restraint of smoothness of the fields $\rho_0(\mathbf{x})$ and $s_0(\mathbf{x})$ is not essential. Consideration of condition (4.5) with discontinuities $[\rho_0] \neq 0$ and $[s_0] \neq 0$ on certain surfaces is carried out as in part 2 and leads to evaluation of (2.6). It should be noted that generalization of the results on the instability of a baroclinic vortex (Sec. 3) to the case of a compressible fluid also leads to (2.6). The corresponding formulation of the problem can be found in [20].

5. Instability of Elastic Bodies. The Lagrange theorem and its inversion are much more important in the theory of elasticity than in hydrodynamics. Let us prove the validity of estimate (2.6) for an elastic body, the linearized equations of motion of which will be written in the form [23]

$$\rho_0\xi_{itt} = \frac{\partial\sigma_{ik}}{\partial x_k},$$
$$\sigma_{ik} = E_{iklm}e_{lm}, \quad 2e_{lm} \equiv \frac{\partial\xi_l}{\partial x_m} + \frac{\partial\xi_m}{\partial x_l}, \tag{5.1}$$

where $\xi$ is the Lagrangian displacement; $E_{ik\ell m}$ is the tensor of the elastic moduli. The general form of Hooke's law is taken for the relationship between the stress tensor $\sigma_{ik}$ and strain tensor $e_{ik}$. The following energy relation is valid for (5.1):

$$E = K + \Pi, \quad \frac{dE}{dt} = \int\limits_{\partial\tau} \xi_{it}\sigma_{ik}n_k dS,$$
$$2K \equiv \int\limits_{\tau} \rho_0\xi_{it}\xi_{it}d\tau, \quad 2\Pi \equiv \int\limits_{\tau} E_{iklm}e_{ik}e_{lm}d\tau, \tag{5.2}$$

from which it is evident that, for energy to be conserved, it is sufficient to require on the boundary of the body $\partial\tau$ that either displacements be absent $\xi = 0$ or that perturbations of the force vector be zero $\sigma_{ik}n_k = 0$.

Let property (2.1) be satisfied for $\Pi$ (5.2). Then, by virtue of the adopted boundary conditions, for the functional (2.2) we again obtain (2.3) and, thus, the estimate (2.6).

It should be noted that an estimate showing that the functional W (2.2) increases linearly with time was obtained for elastic bodies in [5, 6, 24].

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# DEVELOPMENT OF VISCOSITY INSTABILITY IN A POROUS MEDIUM

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The interest in the problem of the stability of two-phase flows undergoing filtration is due mainly to the problem of maximizing the recovery of oil from underground when water or other agents which are immiscible with oil are pumped into the reservoir. When the ratio of the viscosities is large, the displacement of hydrocarbon liquids by water in a porous medium is essentially an unstable process. Instability of the displacement front leads to the formation of "tongues" of liquid which increase in size over time. The linear analysis of stability for piston-like displacement performed in [1] showed that the increase in the amplitude of the tongues is exponential in character. In [2], stability within the framework of a linear approximation was analyzed for the Muskat—Leverett model with allowance for the erosion of the displacement front due to capillary forces. The growth of tongues after loss of stability was analyzed numerically without allowance for capillary forces in [3] for uniform porous media and in [4] for microscopically nonuniform porous media. A detailed analysis of studies of viscosity instability in porous media was given in [5]. At the same time, there has been little study of the stage of nonlinear tongue growth with allowance for the two-phase character of flow behind the displacement front. Here, within the framework of the Baclay—Leverett model, i.e., without allowance for capillary forces, we numerically study the structure of the flow region behind the displacement front in the unstable regime at the nonlinear stage of tongue growth.

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